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Derivatives of Generalized Distance Functions and Existence of Generalized Nearest Points[1]

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The relationship between directional derivatives of generalized distance functions and the existence of generalized nearest points in Banach spaces is investigated. Let G be any nonempty closed subset in a compact locally uniformly convex Banach space. It is proved that if the one-sided directional derivative of the generalized distance function associated to G at x equals to 1 or -1, then the generalized nearest points to x from G exist. We also give a partial answer (Theorem 3.5) to the open problem put forward by S. Fitzpatrick (1989, *Bull. Austral. Math. Soc.* **39**, 233–238). © 2002 Elsevier Science (USA)

1. INTRODUCTION

Let X be a real Banach space of dimension at least 2 and X^* be the dual of X. For a nonempty subset $A \subset X$, as usual, we mean by int A and ∂A the interior and the boundary of A, respectively. We use $B(x, r)$ to denote the closed ball in X with center x and radius $r > 0$. In particular, we write $B = B(0, 1)$.

Throughout this paper, C will denote a closed bounded convex subset of X with $0 \in \text{int}\, C$. Clearly C is an absorbing subset of X but not necessarily

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symmetric. Recall that the Minkowski functional $p_C: X \longrightarrow R$ with respect to the set C is defined by

$$p_C(x) = \inf\{\alpha > 0 : x \in \alpha C\}, \qquad \forall x \in X.$$

For a closed nonempty subset G of X, define the generalized distance function by

$$d_G(x) = \inf_{z \in G} p_C(x - z), \qquad \forall x \in X.$$

A point $z_0 \in G$ with $p_C(x - z_0) = d_G(x)$ is called a generalized nearest point (or generalized best approximation) to x from G. Moreover, for any $x, y \in X$, if the one-sided directional derivative of d_G at x

$$d'_G(x)(y) = \lim_{t \to 0^+} \frac{d_G(x + ty) - d_G(x)}{t}$$

exists, then $-p_C(-y) \leqslant d'_G(x)(y) \leqslant p_C(y)$.

Recently, De Blasi and Myjak [1] and Li [12] investigated the well posedness of generalized best approximation. Their results improve and extend the corresponding results in [2–4, 11, 13, 15].

As shown in [5–8, 10, 14], in the case when $p(\cdot)$ is the norm $\|\cdot\|$, or equivalently, $C = B$, differentiability properties of $d_G(\cdot)$ are related to the existence of the nearest point and continuity of the metric projection P_G which is defined by

$$P_G(x) = \{z \in G : p_C(x - z) = d_G(x)\}.$$

In the present paper, we will investigate the relationship between directional derivatives of generalized distance functions and existence of generalized nearest points in Banach spaces. It is proved that if the one-sided directional derivative of the generalized distance function associated to G at x equals to 1 or -1, then the generalized nearest points to x from G exist provided that X is a compactly locally uniformly convex Banach space. Moreover, we also answer partly the open problem put forward by Fitzpatrick in [9].

2. PRELIMINARIES AND LEMMAS

We first state some well known properties of the Minkowski functional which will be used directly in the rest of the paper, while other properties are referred to [1, 12].

PROPOSITION 2.1. *Let C be as above. Then, for any $x, y \in X$,*

(i) $-p_C(y-x) \leqslant p_C(x) - p_C(y) \leqslant p_C(x-y)$;

(ii) $(p_C(x+ty) - p_C(x))/t \leqslant (p_C(x+t'y) - p_C(x))/t'$, $\forall t, t' \in \mathbf{R}$, $t < t'$;

(iii) $\mu \|x\| \leqslant p_C(x) \leqslant \nu \|x\|$, *where*

$$\mu = \inf_{x \in \partial B} p_C(x) \qquad \textit{and} \qquad \nu = \sup_{x \in \partial B} p_C(x).$$

PROPOSITION 2.2. *Let G be a closed subset of X. Then, for any $x, y \in X$,*

$$-p_C(y-x) \leqslant d_G(x) - d_G(y) \leqslant p_C(x-y)$$

and

$$|d_G(x) - d_G(y)| \leqslant \nu \|x-y\|.$$

DEFINITION 2.1. Let $y \in \partial C$.

(i) C is called compactly locally uniformly convex at y if, for any sequence $\{y_n\} \subset \partial C$, the condition $\lim_{n\to\infty} p_C(y_n + y) = 2$ implies that $\{y_n\}$ has a converging subsequence.

(ii) C is called locally uniformly convex at y if, for any sequence $\{y_n\} \subset \partial C$, the condition $\lim_{n\to\infty} p_C(y_n + y) = 2$ implies that $\lim_{n\to\infty} p_C(y_n - y) = 0$.

(iii) C is called (compactly) locally uniformly convex if C is (compactly) locally uniformly convex at every point $y \in \partial C$.

DEFINITION 2.2. C is called strictly convex if, for any $x, y \in \partial C$, $p_C(x+y) = p_C(x) + p_C(y)$ implies $x = y$.

DEFINITION 2.3. C is called (sequentially) Kadec if, for any sequence $\{x_n\} \subset \partial C$ and $x_0 \in \partial C$, $x_n \to x_0$ weakly implies that $x_n \to x_0$ strongly.

Finally, we still need two lemmas. Recall that a sequence $\{z_n\}$ in G is called a minimizing sequence for $x \in X$ if $\lim_{n\to\infty} p_C(x - z_n) = d_G(x)$.

LEMMA 2.1. *Let G be a closed nonempty subset of X, $x \in X \backslash G$, and $y \in \partial C$. Suppose*

$$\limsup_{t\to 0^+} \frac{d_G(x+ty) - d_G(x)}{t} = 1. \tag{$*$}$$

If $\{z_n\}$ is a minimizing sequence for x, then $\lim_{n\to\infty} p_C(y_n + y) = 2$, where $y_n = (x - z_n)/p_C(x - z_n)$.

Proof. Let $t_n \to 0^+$ such that

$$\lim_{n\to\infty} \frac{d_G(x+t_n y)-d_G(x)}{t_n}=1.$$

We may assume that $0<t_n<d_G(x)\leqslant p_C(x-z_n)$ and $t_n^2>p_C(x-z_n)-d_G(x)$. From Proposition 2.1(ii), it follows that

$$\begin{aligned}\frac{d_G(x+t_n y)-d_G(x)}{t_n} &\leqslant \frac{p_C(x+t_n y-z_n)-p_C(x-z_n)+t_n^2}{t_n}\\ &\leqslant \frac{p_C[(x-z_n)+p_C(x-z_n)\,y]-p_C(x-z_n)}{p_C(x-z_n)}+t_n\\ &= p_C(y_n+y)-1+t_n\end{aligned}$$

and so

$$2\leqslant \liminf_{n\to\infty} p_C(y_n+y)\leqslant \limsup_{n\to\infty} p_C(y_n+y)\leqslant 2.$$

This implies that

$$\lim_{n\to\infty} p_C(y_n+y)=2.$$

The proof is complete. ▮

LEMMA 2.2. *Let* $\{y_n\}\subset \partial C$, $y\in\partial C$ *be such that* $\lim_{n\to\infty} p_C(y_n+y)=2$. *Let*

$$G_0=\left\{z_n=-\left(1+\frac{1}{n}\right)\frac{y_n+y}{p_C(y_n+y)}: n=1,2,\ldots\right\}.$$

Then $d'_{G_0}(0)(y)=1$ *and* $d'_{G_0}(0)(-y)=-1$.

Proof. For every $t>0$, we have

$$d_{G_0}(ty)-d_{G_0}(0)=\inf_n p_C(ty-z_n)-1=\inf_n \left\{p_C(ty-z_n)-p_C(-z_n)+\frac{1}{n}\right\}. \tag{2.1}$$

Let $n_t\in\{1,2,\ldots\}$ be such that

$$\inf_n \left[p_C(ty-z_n)-p_C(-z_n)+\frac{1}{n}\right]\geqslant p_C(ty-z_{n_t})-p_C(-z_{n_t})+\frac{1}{n_t}-t^2. \tag{2.2}$$

Clearly, $\lim_{t\to 0^+} n_t = +\infty$. Let $\alpha_n = (1+1/n)/p_C(y_n+y)$. By Proposition 2.1(ii), we have

$$\frac{p_C[(-t)(-y)-z_n]-p_C(-z_n)}{-t} \leqslant \frac{p_C[\alpha_n(-y)-z_n]-p_C(-z_n)}{\alpha_n}, \qquad \forall t>0. \tag{2.3}$$

Thus, from (2.1)–(2.3) and Proposition 2.2, we have

$$\begin{aligned}
1 = p_C(y) &\geqslant \limsup_{t\to 0^+} \frac{d_{G_0}(ty)-d_{G_0}(0)}{t} \\
&\geqslant \liminf_{t\to 0^+} \frac{d_{G_0}(ty)-d_{G_0}(0)}{t} \\
&\geqslant \liminf_{t\to 0^+} \left(\frac{p_C(ty-z_{n_t})-p_C(-z_{n_t})+\frac{1}{n_t}}{t} - t \right) \\
&\geqslant \liminf_{t\to 0^+} \left(\frac{p_C(ty-z_{n_t})-p_C(-z_{n_t})}{t} - t \right) \\
&\geqslant \liminf_{t\to 0^+} \left(\frac{p_C(-\alpha_{n_t} y-z_{n_t})-p_C(-z_{n_t})}{-\alpha_{n_t}} - t \right) \\
&\geqslant \liminf_{t\to 0^+} \left(\frac{p_C(-\alpha_{n_t} y-z_{n_t})-p_C(-z_{n_t})}{-\alpha_{n_t}} \right) + \liminf_{t\to 0^+} (-t) \\
&= \liminf_{n_t\to\infty} \frac{p_C[-\alpha_{n_t} y+\alpha_{n_t}(y_{n_t}+y)]-p_C[\alpha_{n_t}(y_{n_t}+y)]}{-\alpha_{n_t}} \\
&= \liminf_{n_t\to\infty} (-p_C(y_{n_t})+p_C(y_{n_t}+y)) = 2-1 = 1
\end{aligned}$$

so that $d'_G(0)(y) = 1$.

By a similar argument one can show $d'_{G_0}(0)(-y) = -1$ and the proof is complete. ▮

Remark 2.1. In the case when $p_C(\cdot)$ is a norm, $d'_{G_0}(0)(-y) = -1$ was proved by Fitzpatrick in [9]. He also mentioned that $d'_{G_0}(0)(y) = 1$ with no proof. We note that the fact $d'_{G_0}(0)(y) = 1$ is not trivial as $d'_{G_0}(0)(y)$ is not homogenous in y, in general, so that it can not be deduced directly from $d'_{G_0}(0)(-y) = -1$.

3. MAIN RESULTS

Before proving the main theorems, we introduce the concept of approximative compactness of G for $x \in X$.

DEFINITION 3.1. Let G be a closed nonempty subset of a Banach space X and $x \in X$. G is called approximatively compact for x if any minimizing sequence for x has converging subsequences.

THEOREM 3.1. *Let $y \in \partial C$. Then the following statements are equivalent:*

(i) *for any nonempty closed subset G of X and $x \in X \backslash G$, if* (*) *is satisfied, then G is approximatively compact for x;*

(ii) *for any nonempty closed subset G of X and $x \in X \backslash G$, if $d'_G(x)(y) = 1$, then G is approximatively compact for x;*

(iii) *C is compactly locally uniformly convex at y.*

Proof. (i) $\Rightarrow$ (ii). It is obvious.

(ii) $\Rightarrow$ (iii). Suppose (iii) does not hold. Then there exists a sequence $\{y_n\} \subset \partial C$ such that $\lim_{n \to \infty} p_C(y_n + y) = 2$ but $\{y_n\}$ has no converging subsequences. Let

$$G = \left\{ x - \left(1 + \frac{1}{n}\right) \frac{y_n + y}{p_C(y_n + y)} : n = 1, 2, \ldots \right\}.$$

Then G is closed and

$$d_G(x) = \inf_{z \in G} p_C(x - z) = 1.$$

Note that, for any $z \in G$, $z = x - (1 + \frac{1}{n})((y_n + y)/p_C(y_n + y))$ for some $n \geqslant 1$. Thus,

$$p_C(x - z) = 1 + \frac{1}{n} > d_G(x)$$

so that x has no nearest point in G. However, from Lemma 2.2 we have that $d'_G(x)(y) = d'_{G_0}(0)(y) = 1$, which contradicts to (ii).

(iii) $\Rightarrow$ (i). Assume that (iii) holds and $x \in X \backslash G$ satisfies (*). Let $\{z_n\}$ be any minimizing sequence for x and $y_n = (x - z_n)/p_C(x - z_n)$. Then by virtue of Lemma 2.1, we have $\lim_{n \to \infty} p_C(y_n + y) = 2$. Since C is compactly locally uniformly convex at y, $\{y_n\}$ has a converging subsequence. Consequently, $\{z_n\}$ has a converging subsequence. The proof is complete. ▌

COROLLARY 3.1. *The following statements are equivalent :*

(i) *for each closed nonempty subset G of X and $x \in X \backslash G$, if there is $y \in \partial C$ with $d'_G(x)(y) = 1$, then G is approximatively compact for x;*

(ii) *C is compactly locally uniformly convex.*

THEOREM 3.2. *Let $y \in \partial C$. The following statements are equivalent:*

(i) *for each nonempty closed subset G of X and $x \in X \setminus G$, if (*) holds, then G is approximatively compact for x and $P_G(x) = x - d_G(x)\, y$;*

(ii) *for each nonempty closed subset G of X and $x \in X \setminus G$, if $d'_G(x)(y) = 1$, then G is approximatively compact for x and $P_G(x) = x - d_G(x)\, y$;*

(iii) *C is locally uniformly convex at y.*

Proof. (i) $\Rightarrow$ (ii). It is obvious.

(ii) $\Rightarrow$ (iii). Suppose C is not locally uniformly convex at y. Then there is a sequence $\{y_n\} \subset \partial C$ such that

$$\lim_{n \to \infty} p_C(y_n + y) = 2 \qquad \text{and} \qquad p_C(y_n - y) \geqslant \delta > 0$$

for all n. By virtue of Theorem 3.1, $\{y_n\}$ has a converging subsequence, denoted by $\{y_n\}$ itself. Let $y_0 \in \partial C$ be such that $p_C(y_n - y_0) \to 0$. Clearly, $p_C(y_0 - y) \geqslant \delta$ and $p_C(y_0 + y) = 2$. Let

$$G = \{x - y, x - y_0\}.$$

Then $p_C(ty + y_0) = 1 + t$ for each $t > 0$. Indeed, choose $x^* \in X^*$ with $\sup_{x \in C} \langle x^*, x \rangle \leqslant 1$ such that

$$\left\langle x^*, \frac{y_0 + y}{2} \right\rangle = p_C\left(\frac{y_0 + y}{2}\right) = 1$$

and so $\langle x^*, y_0 \rangle = \langle x^*, y \rangle = 1$. It follows that

$$1 + t = tp_C(y) + p_C(y_0) \geqslant p_C(ty + y_0) \geqslant \langle x^*, ty + y_0 \rangle = 1 + t.$$

Hence,

$$p_C[(x + ty) - (x - y_0)] = p_C(ty + y_0) = 1 + t$$

and

$$p_C[(x + ty) - (x - y)] = p_C((t + 1)\, y) = 1 + t$$

so that $d_G(x + ty) = 1 + t$, $d_G(x) = 1$, which implies that $d'_G(x)(y) = 1$. However, $P_G(x) = \{x - y, x - y_0\} = G$, contradicting (ii).

(iii) $\Rightarrow$ (i). Suppose that $x \in X \setminus G$ and (*) holds. By Lemma 2.1, it follows that any minimizing sequence $\{z_n\}$ for x satisfies that $\lim_{n \to \infty} p_C(y_n + y) = 2$, where $y_n = (x - z_n)/p_C(x - z_n)$. Observe that C is locally uniformly convex at y. We have $\lim_{n \to \infty} p_C(y_n - y) = 0$ so that $z_n \to x - d_G(x)\, y$ since $\lim_{n \to \infty} p_C(x - z_n) = d_G(x)$. Therefore (i) holds and the proof is complete. ∎

COROLLARY 3.2. *The following statements are equivalent:*

(i) *for each closed nonempty subset G of X and $x \in X \setminus G$, if there is $y \in \partial C$ such that $d'_G(x)(y) = 1$, then G is approximatively compact for x and $P_G(x) = x - d_G(x)\, y$;*

(ii) *C is locally uniformly convex.*

THEOREM 3.3. *Let $-y \in \partial C$. The following statements are equivalent:*

(i) *for each nonempty closed subset G of X and $x \in X \setminus G$, if $\liminf_{t \to 0^+} ((d_G(x+ty) - d_G(x))/t) = -1$, then $P_G(x) \neq \emptyset$;*

(ii) *for each nonempty closed subset G of X and $x \in X \setminus G$, if $d'_G(x)(y) = -1$, then $P_G(x) \neq \emptyset$;*

(iii) *C is compactly locally uniformly convex at $-y$.*

Proof. (i) $\Rightarrow$ (ii). It is obvious.

(ii) $\Rightarrow$ (iii). Suppose C is not compactly locally uniformly convex at $-y$. Then there is a sequence $\{y_n\} \subset \partial C$ such that $\lim_{n\to\infty} p_C(y_n - y) = 2$, but $\{y_n\}$ has no converging subsequences. Set

$$G' = \left\{ x - \left(1 + \frac{1}{n}\right) \frac{y_n - y}{p_C(y_n - y)} : n = 1, 2, \ldots \right\}.$$

Then G' is closed. Exactly as in the proof of (ii) $\Rightarrow$ (iii) of Theorem 3.1, we can show $P_{G'}(x) = \emptyset$. However, by Lemma 2.2, $d'_{G'}(x)(y) = -1$, contradicting (ii).

(iii) $\Rightarrow$ (i). Let $t_n \to 0^+$ satisfy $\lim_{n\to\infty} ((d_G(x + t_n y) - d_G(x))/t_n) = -1$. Select $\{z_n\} \subset G$ such that

$$p_C(x + t_n y - z_n) < d_G(x + t_n y) + t_n^2.$$

Then, by Proposition 2.1(ii), we obtain

$$\frac{p_C(x + t_n y - z_n) - p_C(x - z_n)}{t_n} \geqslant \frac{p_C[x - p_C(x - z_n)\, y - z_n] - p_C(x - z_n)}{-p_C(x - z_n)}$$

and so

$$\begin{aligned} \frac{d_G(x + t_n y) - d_G(x)}{t_n} &\geqslant \frac{p_C(x + t_n y - z_n) - p_C(x - z_n)}{t_n} - t_n \\ &\geqslant -t_n + \frac{p_C(x - z_n) - p_C[(x - z_n) - p_C(x - z_n)\, y]}{p_C(x - z_n)} \\ &= -t_n + 1 - p_C(y_n - y), \end{aligned}$$

where $y_n = (x - z_n)/p_C(x - z_n)$. Thus, $\{y_n\} \subset \partial C$ and $\lim_{n\to\infty} p_C(y_n - y) = 2$, which implies that $\{y_n\}$ has a converging subsequence, denoted by $\{y_{n_k}\}$. Using Proposition 2.1(iii) and 2.2, we have

$$\begin{aligned} d_G(x) &\leqslant p_C(x - z_n) \\ &\leqslant p_C(x + t_n y - z_n) + p_C(-t_n y) \\ &\leqslant (d_G(x + t_n y) + t_n^2) + t_n \\ &\leqslant d_G(x) + v t_n \|y\| + t_n^2 + t_n. \end{aligned}$$

Hence,

$$\lim_{n\to\infty} p_C(x - z_n) = d_G(x).$$

Consequently, $\{z_{n_k}\}$ converges to some point z_0 and $p_C(x - z_0) = d_G(x)$. This implies that (i) holds and the proof is complete. ∎

COROLLARY 3.3. *The following statements are equivalent:*

(i) *for each closed nonempty subset G of X and $x \in X \setminus G$, there is $-y \in \partial C$ with $d'_G(x)(y) = -1$, if and only if $P_G(x) \neq \emptyset$;*

(ii) *C is compactly locally uniformly convex.*

Proof. By virtue of Theorem 3.3, it suffices to show that there is $-y \in \partial C$ with $d'_G(x)(y) = -1$ if $P_G(x) \neq \emptyset$. For this purpose, take $g_0 \in P_G(x)$. Then $g_0 \in P_G(g_0 + t(x - g_0))$ for every $t \in (0, 1]$. Put $y = (g_0 - x)/p_C(x - g_0)$. We have $-y \in \partial C$ and

$$d_G(x + ty) = p_C(x - g_0) - t, \qquad \forall t \in [0, 1].$$

Thus

$$d'_G(x)(y) = \lim_{t\to 0^+} \frac{p_C(x - g_0) - t - p_C(x - g_0)}{t} = -1,$$

completing the proof. ∎

COROLLARY 3.4. *Let G be a closed nonempty subset and C a compactly locally uniformly convex subset of a Banach space X. Then G is proximinal (i.e., $P_G(x) \neq \emptyset$ for every $x \in X$) if and only if, for every $x \in X \setminus G$, there is $-y \in \partial C$ satisfying $d'_G(x)(y) = -1$.*

Remark 3.1. In the case when $p_C(\cdot) = \|\cdot\|$, it is easy to show that C is compactly locally uniformly convex at $-y \in \partial C$ if and only if C is compactly locally uniformly convex at $y \in \partial C$.

Remark 3.2. It is noted that, in Theorem 3.3 and Corollary 3.3, the conclusion that $P_G(x) \neq \emptyset$ can not be improved to the stronger one that G is approximatively compact for x. Similarly, in Corollary 3.4, the conclusion that G is proximinal can not be improved to the stronger one that G is approximatively compact. Furthermore, even in the case when C is locally uniformly convex, one can not deduce that $P_G(x)$ is a singleton from $d'_G(x)(y) = -1$, where $-y \in \partial C$. For example, let X be an arbitrary locally uniformly convex Banach space of infinite dimension and C the closed unit ball of X. Define $G = \{x \in X : \|x\| \geqslant 1\}$. Obviously, G is proximinal. Thus, for any $x \in X \setminus G$, there is $-y \in \partial C$ such that $d'_G(x)(y) = -1$ by Corollary 3.4. However, G is neither a Chebyshev set nor an approximatively compact set.

THEOREM 3.4. *Let G be a nonempty closed subset of Banach space X and $x \in X \setminus G$. If G is approximatively compact for x and $P_G(x)$ is a singleton, then there exists $y \in \partial C$ satisfying $d'_G(x)(y) = 1$.*

Proof. Let $P_G(x) = \{g_0\}$ and $x_t = x + t(x - g_0)$, $t \in (0, 1)$. By virtue of the definition of $d_G(x_t)$, there is $g_t \in G$ satisfying $p_C(x_t - g_t) < d_G(x_t) + t^2$. Clearly, $x_t \to x$ as $t \to 0^+$ and $p_C(x - g_t) - p_C(x - g_0) \geqslant 0$. Choose $x_t^* \in X^*$ such that $\sup_{x \in C} \langle x^*, x\rangle = 1$ and $\langle x_t^*, x - g_t\rangle = p_C(x - g_t)$. Then,

$$
\begin{aligned}
&\frac{d_G(x_t) - d_G(x)}{t} - p_C(x - g_0)\\
&\geqslant \frac{p_C(x_t - g_t) - p_C(x - g_0)}{t} - p_C(x - g_0) - t\\
&\geqslant \frac{\langle x_t^*, x_t - g_t\rangle - p_C(x - g_0)}{t} - p_C(x - g_0) - t\\
&= \frac{\langle x_t^*, x_t - x\rangle}{t} + \frac{p_C(x - g_t) - p_C(x - g_0)}{t} - p_C(x - g_0) - t\\
&\geqslant \langle x_t^*, x - g_0\rangle - p_C(x - g_0) - t\\
&= (\langle x_t^*, x - g_t\rangle + \langle x_t^*, g_t - g_0\rangle) - p_C(x - g_0) - t\\
&\geqslant (p_C(x - g_t) - p_C(x - g_0)) - p_C(g_0 - g_t) - t\\
&\geqslant -p_C(g_0 - g_t) - t.
\end{aligned}
$$

Thus

$$\frac{d_G(x_t) - d_G(x)}{t} \geqslant p_C(x - g_0) - p_C(g_0 - g_t) - t. \tag{3.1}$$

Using Propositions 2.1(iii) and 2.2, we get

$$\begin{aligned} d_G(x) &\leqslant p_C(x-g_t) \\ &= p_C[x_t - t(x-g_0) - g_t] \\ &\leqslant p_C(x_t - g_t) + t p_C[-(x-g_0)] \\ &\leqslant d_G(x_t) + t^2 + tv\,\|x-g_0\| \\ &\leqslant d_G(x) + 2vt\,\|x-g_0\| + t^2. \end{aligned}$$

It follows that

$$\lim_{t\to 0^+} p_C(x-g_t) = d_G(x) = p_C(x-g_0) > 0.$$

Since G is approximatively compact for x and $P_G(x)=\{g_0\}$, $\lim_{t\to 0^+} p_C(g_0-g_t)=0$. This with (3.1) implies that

$$\liminf_{t\to 0^+} \frac{d_G(x_t)-d_G(x)}{t} \geqslant p_C(x-g_0).$$

Obviously,

$$\limsup_{t\to 0^+} \frac{d_G(x_t)-d_G(x)}{t} \leqslant p_C(x-g_0).$$

Hence $d'_G(x)(x-g_0)=p_C(x-g_0)$. By the positive homogeneity of $d'_G(x)(u)$ with respect to u, we have $d'_G(x)(y)=1$ with $y=(x-g_0)/p_C(x-g_0)$. This completes the proof. ∎

COROLLARY 3.5. *Let C be locally uniformly convex and G nonempty closed subset of X. Then for every $x\in X\setminus G$ there exists a point $y\in\partial C$ such that $d'_G(x)(y)=1$ if and only if G is an approximatively compact Chebyshev subset of X.*

Proof. This result follows from Corollary 3.2 and Theorem 3.4. ∎

THEOREM 3.5. *Let G be a nonempty closed subset of X. If X is reflexive and C is both strictly convex and Kadec, then the set*

$$\mathscr{D} = \{x\in X\setminus G;\ \exists y\in\partial C \text{ with } d'_G(x)(y)=1\}$$

is residual in $X\setminus G$.

Proof. Let $X_0(G)$ be the set of all points $x\in X\setminus G$ such that the problem $\min_C(x, G)$ is well posed, by which we mean that G is approximatively compact for x and $P_G(x)$ is a singleton. Thus, for each point $x \in X_0(G)$, there exists $y \in \partial C$ such that $d'_G(x)(y) = 1$ from Theorem 3.4. This implies that $X_0(G) \subset \mathscr{D}$. From Theorem 3.3 in [12], $X_0(G)$ is residual in $X \setminus G$, so is $\mathscr{D}$. ▮

Remark 3.3. In the case when $p(\cdot) = \|\cdot\|$, Fitzpatrick [9] put forward the following open problem: If G is a closed subset of reflexive Banach space X, is the set $\mathscr{D}$ residual in $X \setminus G$? Clearly, Theorem 3.5 gives an affirmative answer to the problem under the assumption that C is both strictly convex and Kadec. We do not know whether Theorem 3.5 remains true without this assumption.

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